Deciding Vote Percentage

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Deciding Vote Percentage is a new metric for measuring partisan gerrymandering. It is defined as the probability, under a given election map, of a particular person casting the deciding vote in an election. It represents the value to the voter of participating in the election and, unsurprisingly, decreases under gerrymandered maps.

This document is incomplete. I wanted to get this idea out there quickly and this document is missing some of the expected parts of an academic paper. This document does contain the legal basis for the measure, its definition, and a few rudimentary examples of its application. The document does not contain any application of the idea to "real world" data. The document ends with review of the legal arguments and a conclusion.

1 Legal

This section covers some of the legal issues with gerrymandering, especially ones that influenced my choice of metric. To be clear, I am not a lawyer. I do programming, math and finance for a living. (Although, I do read Supreme Court decisions for fun.)

The U.S. Constitution says the state legislatures decide the "Manner of holding Elections". If there are districts, the state legislature gets to set them. Thus, partisan gerrymandering seems to be completely legal. Our best hope is to turn the constitution against itself. The constitution says other things about elections, so we have to show that when a state legislature does partisan gerrymandering, it is contradicting something else in the U.S. Constitution. Unfortunately, the constitution says very little about elections.

The constitution says:

- Article I, Section 2: "The House of Representatives shall be composed of members chosen every second Year by the People of the several States,..."
- Article I, Section 2: Hold a census every ten years, and then reapportion the number of Representatives to each state.

- Article I, Section 4: "The Times, Places and Manner of holding Elections for Senators and Representatives, shall be prescribed in each State by the Legislature thereof; but the Congress may at any time by Law make or alter such Regulations, except as to the Places of chusing Senators."
- Article IV, Section 4: "The United States shall guarantee to every State in the Union a Republican Form of Government"
- 14th Amendment: "nor shall any State ... deny to any person within its jurisdiction the equal protection of the laws"
- 14th Amendment: *from Baker v. Carr* One person, one vote.
- 15th Amendments: "The right of citizens of the United States to vote shall not be denied or abridged by the United States or by any State on account of race, color, or previous condition of servitude."
- 19th Amendment: "The right of citizens of the United States to vote shall not be denied or abridged by the United States or by any State on account of sex."
- 26th Amendment: "The right of citizens of the United States, who are 18 years of age or older, to vote, shall not be denied or abridged by the United States or any state on account of age."

The constitution gives us very little to work with. The most obvious target is "one person, one vote". I use that one for my metric, which I present later in this document.

My metric can make a second argument based on the fact that the constitution requires an election (Article I, Section 4). It would be absurd for the courts to allow an "election" where there is only one candidate. But, thanks to gerrymandering, many one-candidate elections happen because the other party cannot find a candidate to run in the district. Thus, a constitution argument can be made against sham elections, where voters have no real power.

Another likely target for a metric is the phrase "every second Year". Politicians gerrymander so they have an easy re-election. They do not want the threat of being voted out until the next reapportionment, 10 years in the future. So, a possible legal approach would be to say that gerrymanderers are probabilisticly violating the call for an election every 2 years.

A twist on the "every second Year" approach would be to say that it is not a case of gerrymandering to benefit a party but to benefit *all incumbents*. In partisan gerrymandering, minority districts are "packed" and even less likely to flip to the other party than "cracked" districts. So, all districts favor the incumbent. The courts may be more receptive to this argument than a partisan one.

1.1 Parties

The constitution says nothing about parties. Justices have, on occasion, been welcoming to the idea that parties live in the "freedom of association" that is part of the rights to free speech and right to assemble in the First Amendment.

While courts have recognized parties, they have not recognized any right to proportional representation for parties. If your party wins 70% of the vote, it does not mean your party gets 70% of the Representatives.

The courts have recognized, especially at the state legislature level, that it is wrong for a party to get less than half of the votes and get more than half of the legislators. Given our two party system, the real power comes from having a majority of legislators and being able to control the positions like Speaker of the House, control the legislation that comes to the floor, and have a strong chance of passing that legislation. The courts see that a party with the minority of votes having that real power is unconstitutional.

1.2 Requirements

When designing a metric, it is important to remember that the legal requirement for gerrymandering is equivalent to that of murder. For murder, we need a bad guy, the intent to kill the victim, and the victim killed by the bad guy. So, for gerrymandering, we need the politicians doing the redistricting, the intent to partisan gerrymander, and a victim hurt by gerrymandering. It is very easy to get the first two. (The courts think any competent politician intends to gerrymander.) The last one is the hard part.

The legal requirement is *not* attempted murder. Attempted murder needs a bad guy, the intent to kill, and an action intended to kill the victim. In gerrymandering, we have a lot of attempted gerrymandering: politicians doing the redistricting, an intent to gerrymander, and actions done by those politicians with the hope of gerrymandering. But that is not enough. The court needs to see that someone was harmed by gerrymandering. At times, it feels like the courts do not think the gerrymanderers know what they are doing, when they have been doing it for more than 100 years!

For example, in the Whitford v. Gill case, we have communications from Wisconsin legislators conspiring with a mathematician from Oklahoma. If these people were in any way competent, there are millions of victims. But the court requires finding a specific one and showing a specific harm. The harm is often hard to formulate. One of the strengths of the Efficiency Gap's "wasted votes" is that it points to concrete victims.

2 A New Metric

I am proposing a new metric to measure partisan gerrymandering. The "Deciding Vote Percentage" is the probability that an individual voter casts the deciding vote in the election for a legislature. That is, the probability that their vote will decide the majority party in the legislature. This is a simple metric of each voter's power. As you will see, gerrymandering dramatically lowers it.

To show partisan gerrymandering in court, the plaintiff can show that gerrymandering lowers the voters' power unevenly, creating a "one person, one vote" argument for the court. Another approach is to create an alternative election map with dramatically higher voter power and convince the courts that the voting power in the actual election map is insignificant and represents a sham election.

I will work through some examples of applying the metric and, afterwards, discuss the legal application of it.

2.1 First Example

In the simplest example, we have a state legislature with 3 seats. Each seat has a district with 5 voters. Voters choose between only 2 political parties: Orange and Purple.

Obviously, a party controls the state legislature if they hold 2 of the 3 seats. A voter casts the deciding vote if everything else is tied. That is, both (1) the two districts without the voter go for different parties and (2) the four other voters in their district split equally for different parties.

Now, in order to measure the probability of an individual casting the deciding vote, we need a model that predicts the behavior of other voters. This is a weakness of this metric. The courts do not like predictions of voter behavior. And the 2016 presidential election showed how bad our country can be at predicting elections. Nonetheless, those who do gerrymandering use a prediction of voter behavior and I think the harm of gerrymandering starts when the districts are set, not at the election. Thus, my metric uses a prediction model.

There are 15 voters. For the purposes of predicting, let's say 8 voted for the Orange party in the last election and 7 voted for the Purple party, and let's say there is a 75% chance that they will vote for the same party in the next election and a 25% chance they will change their vote.

If those voters are evenly distributed, we will have 2 districts with 3 past Orange voters and 2 past Purple voters ("3O2P"), and 1 district with 2 past Orange voters and 3 past Purple voters ("2O3P"). The 3O2P districts have a 61.5% chance of electing an Orange representative and 38.5% chance of electing a Purple. And vice versa for the 2O3P district.

Now we can compute the deciding vote percentage of a past Orange voter in a 3O2P district. They will only cast the deciding vote if the 3O2P and 2O3P districts go for opposing parties. That probability is $61.5\% \cdot 61.5\% + 38.5\% \cdot 38.5\% = 52.7\%$. But for the past Orange voter to cast the deciding vote, the other 4 voters in his district must go evenly for different parties. That probability is 46.1%. So, the deciding vote percentage for that voter is $52.7\% \cdot 46.1\%$, which is 24.3%.

That value, 24.3%, is the metric's value for that voter. It is a measure of their power, with larger values being better. 24.3% is actually the value for 6 voters, since there are 6 voters that have voted in the past for Orange in 3O2P districts. It is okay for the numbers for all voters to sum to over 100%, because it is possible for multiple voters to be the decisive voter at the same time. The values for the other voters are in the table below.

District	Past vote	Other districts tie	Other voters tie	Deciding Vote %age
3O2P	Orange	52.7%	46.1%	24.3%
3O2P	Purple	52.7%	35.2%	18.5%
2O3P	Orange	47.3%	35.2%	16.6%
2O3P	Purple	47.3%	46.1%	21.8%

The minimum deciding vote percentage for any voter is 16.6% and it occurs for the past Orange voter in the Purple-majority district. They suffer because it is both unlikely that the 3O2P districts go different directions and unlikely that the other voters in the district end up in a tie.

Now, consider a gerrymandered situation. The Orange party tries to solidify its position by creating two districts with 4 Orange and 1 Purple ("4O1P") and then a packed district with 5 Purple ("5P").

The 4O1P has a 79.1% chance of electing an Orange candidate and a 20.9% chance of going Purple. The 5P has 10.4% of voting Orange and a 89.6% of voting Purple.

For an Orange voter in a 4O1P district, the chance of the other two districts going for different parties is 73.0%. That high value is not surprising when one is gerrymandered to be an Orange district and the other to be a Purple one. The chance that the other voters split equally is 35.2%. Thus, the deciding vote percentage is $73.0\% \cdot 35.2\% = 25.7\%$.

District	Past vote	Other districts tie	Other voters tie	Deciding Vote %age
4O1P	Orange	73.0%	35.2%	25.7%
4O1P	Purple	73.0%	21.1%	15.4%
5P	Purple	33.1%	21.1%	7.0%

Looking at the table of voters, we see some voters are losing power. The worst off are the Purple voters in the Purple packed district who go from at least 18.5% under fair districts to 7.0% under the gerrymandered — cut in more than half. The other Purple voters are also worse off under gerrymandering. The past Orange voters gain a little power, but that is not always the case. It is possible under gerrymandering for all voters to decrease in deciding vote percentage.

All partys' power can decrease for two reasons. In situations with more than 3 districts, we would expect the number of cracked districts to be larger, which decreases the "other districts tie" part of the power metric. In real cases, you would also expect the margin in each district to be higher, which decreases both parts of the power metric. These decreases happens to voters in the opposing party and to voters in the gerrymandering party.

We can also look at the sum total of deciding vote percentage. That is, adding up the probability that each voter will cast the deciding vote. With fair districting, the total was 318.4%. With gerrymandering, the total is 271.4%. Gerrymandering decreases the probability of ties and, therefore, decreases the probability that any voter will cast a deciding vote. This, more than anything, is the insidiousness of gerrymandering.

This simple example had an odd number of districts and an odd number of voters in each district; I chose odd numbers to avoid ties. Even numbers can be handled, by assuming that tied votes result in power sharing between the parties. Thus, the deciding vote would create or break a tie and is worth half as much. Another way to see it is that to switch the majority from one party to the other requires two deciding votes, one to create the tie and then another to break it for the opposition party, so the two deciding voters each get half the credit.

The deciding vote percentage can be adapted to a number of difficult situations. It can handle more than 2 parties, possibly with some complexity. It is easy to handle the case common in the U.S., where it is impossible for the third parties to have the majority in the legislature. The deciding vote percentage can also handle when someone in the last election ran unopposed. The prediction model will have more error, but it can be done.

2.2 Second Example

Now, consider a similar example with 3 districts, but a large number of voters. Let the number of voters be $6N + 3$, so that each district has $2N + 1$ voters. We will assume they are nearly evenly split, with $3N + 2$ of them having voted for Orange in the past and $3N + 1$ having voted for Purple. Again, we assume that if a voter voted for a party in the past, there is a 75% chance they will vote for the same party in the future.

Given the large number N, the distribution of votes can be approximated by a normal distribution. If we are counting the votes for Orange, the normal distribution is centered on the expected number of votes for that party. That value would be .75 times the number of past Orange voters plus .25 times the number of past Purple voters. The normal distribution's standard deviation is estimated from the binomial distribution's for each group of voters. The binomial distribution's formula for variance is $np(1 - p)$, for n voters and probability p. If o is the number of past Orange voters, the variance due to past Orange voters is $o \cdot 75\% \cdot .25\%$ and the variance due to past Purple voters is $(2N + 1 - o) \cdot 25\% \cdot 75\%$. Adding the variances of the two groups yields $\sigma^2 = (2N + 1) \cdot 75\% \cdot 25\%$. Notice that the total variance does not depend on the number of past Orange voters, only the total number of voters, which is the same for every district.

With voters split nearly equally in each district, the probability of the other two districts going for opposite parties is 50% for large N. The probability of a tie between voters in a district is the probability that the other $2N$ voters have exactly N votes for Orange. Since N is the expected number of votes for Orange, it is at the center of the normal distribution. Therefore, the probability of a tie can be estimated by $\phi(0)/\sigma$ for large values of N, where ϕ is the probability density function of the normal distribution.

If $2N + 1 = 167637$, as in the case of districts for the Texas House of Representatives, then the probability of a tie vote in a district is 0.22%. Multiplying by the chance of districts tying yields the deciding vote percentage of 0.11%. This value is pretty small, but rather large considering there is more than half a million people voting. The sum total of deciding vote percentage is 566%.

Now, the gerrymandered case. Let's say Orange does the redistricting and creates two "cracked" districts with each one having $N+1+K$ past Orange voters and $N-K$ past Purple voters. The remaining "packed" district has $N - 2K$ past Orange voters and $N + 1 + 2K$ past Purple voters.

The cracked district has an expected number of Orange votes of $.75(N + 1 + K) + .25(N - K) = .75 + N + .5K$. For simplicity, we will assume $K \gg .75$ and drop the .75. The standard deviation, σ does not change with K, since σ only depends on the number of people in the district and that is not affected by K. The probability that Orange wins the vote in the cracked district is estimated by $\Phi((N + .5K - N)/\sigma) = \Phi(.5K/\sigma)$, where Φ is the cumulative distribution function for the normal distribution. Similarly, the probability that Orange wins the packed district is $\Phi(-K/\sigma)$.

For a past Orange voter in a cracked district, the probability of the other two districts going for opposite parties is:

$$\Phi(.5K/\sigma)(1 - \Phi(-K/\sigma)) + (1 - \Phi(.5K/\sigma))\Phi(-K/\sigma)$$

The probability of a tie vote in the cracked district is $\phi(.5K/\sigma)/\sigma$. So, the formula for the deciding vote percentage is:

$$(\Phi(.5K/\sigma)(1 - \Phi(-K/\sigma)) + (1 - \Phi(.5K/\sigma))\Phi(-K/\sigma))\phi(.5K/\sigma)/\sigma$$

For the past Purple voter in a cracked district, the deciding vote percentage is the same value. (This is a result of approximating $K + 1$ with K.)

For the voters in the packed district, the deciding vote percentage is:

$$(\Phi(.5K/\sigma)(1 - \Phi(.5K/\sigma)) + (1 - \Phi(.5K/\sigma))\Phi(.5K/\sigma))\phi(-K/\sigma)/\sigma$$

To better interpret these equations, I ran them in Gnumeric with $2N+1 = 167637$ and varied the value for K. The results are in following table.

K	Orange Win	DV% in Cracked Dist.	DV% in Packed Dist.	Sum DV%
$0\sigma = 0$	50.0%	0.11%	0.11%	565%
$2.2\sigma = 390$	75.0%	0.10%	0.0047%	360%
$4.0\sigma = 709$	95.0%	0.029%	0.0000034%	100%
$5.2\sigma = 922$	99.0%	0.0076%	0.0000000028%	26%
$6.6\sigma = 1170$	99.9%	0.00097%	0.000000000000075%	4%

For $K = 390$, Orange has a 75% chance of winning the election. The deciding vote percentage for voters in the cracked district has dropped slightly from 0.11% to 0.10%. The deciding vote percentage for voters in the packed district has gone from 0.11% to 0.0047%. Those voters now have 1/24th the power they had in a fair map. And that is only for exchanging 780 pairs of voters out of 502125 voters total.

For $K = 709$, Orange has a 95% chance of winning the election. Voters in the cracked districts now have a deciding vote percentage of 0.029%, roughly a quarter of what they had with a fair map. The voters in the packed district have 0.0000034%. A minuscule amount. When people in gerrymandered districts say "it doesn't matter if I vote", this is what they are talking about. The Orange party achieves this by moving less than 1% of the voters between districts.

For $K = 922$, Orange has a 99% chance of winning the election. Voters in the cracked districts have a deciding vote percentage of 0.0076%, or roughly 1/15th their power with a fair map. Voters in the packed district have 0.0000000028%, or 1-over-40-million of their power with a fair map. It only takes a small amount of gerrymandering to make everyone's vote worth a fraction of what it could be.

The voters in the cracked districts lose less power for two reasons. First, the other two districts are very likely to go for opposite parties. They are designed to. Second, the cracked districts have half the margin seen in the packed districts. The power of voters in cracked districts actually increases for small values of K, but for $K > 250$, it decreases with larger K. And it does down quickly. For $K = 1170$, their deciding vote percentage is 0.00097%, or less than a hundredth of that of a fair map.

The voters in the packed district have it worse. The other districts are both designed to go Orange, so there is very little chance that they go for opposite parties. Moreover, the margin between past Orange and past Purple voters is larger in the packed district. It only takes $K = 471$ for them to be reduced to a deciding vote percentage less than a hundredth of what they would have in a fair map. Just from swapping 942 pairs of voters out of the 502125 voters total.

The sum of all voters' deciding vote percentage drops as K increases. It is almost linear in relation to the probability that Orange loses. This is because as the probability of Orange loses goes to 0%, the chances of a tie gets smaller as well.

2.3 Real World Data

Unfortunately, this document is being released early, before it can be applied to real-world data. I expect that other researchers or activists will have a chance to apply the metric before I will. In fact, I'm releasing this early in the hopes they will.

3 Commentary

There are two possible argument to make in court. The first uses the 14th Amendment. To apply the metric, the plaintiff requires a predictive model of voter behavior and the existing district map. Voting power in packed districts should be dramatically lower than those in cracked districts. The courts should accept a "one person, one vote" argument based on the 14th Amendment.

The second argument is more complicated to make. It requires the predictive model, existing district map, and a proposed alternative district map. The quality each maps will be determined by the minimum deciding vote percentage of all voters in the map. If the alternative district map has a much higher minimum deciding vote percentage than the existing map, the courts can conclude that the legislature did not give the voter one vote, but a fraction of a vote. That is, it was a sham election, which would violate Article I, Section 4.

Courts will find weaknesses with the deciding vote percentage. It relies a model that predict voters' behavior. I think that is fine, but courts have been skeptical. Another weakness is that it does not give an absolute scale where 1.0 is definitely gerrymandered and 0.0 is not. It creates a set of numbers for a given district map and the result requires some interpretation.

4 Conclusion

I have proposed a new metric, Deciding Vote Percentage, to measure partisan gerrymandering. It produces a number that measures the voter's power in an election. It was crafted to be compatible with legal precedent. It can be used to make a "one person, one vote" argument or a sham election argument to say there are limits on how state legislatures can set districts for elections.